# Bounded Gain of Energy on the Breathing Circle Billiard

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**Abstract**

The Breathing Circle is a 2-dimensional generalization of the Fermi Accelerator. It is shown that the billiard map associated to this model has invariant curves in phase space, implying that any particle will have bounded gain of energy.

## 1 Introduction

The breathing circle is the time-dependent plane region bounded, instantaneously, by a circle whose radius is a periodic function of time. The billiard problem on the breathing circle consists on the free motion of a point particle inside this region, colliding elastically with the moving boundary.

As J. Koiller, R. Markarian and ourselves have showed in [5], time varying billiard computations can be performed in the same way as done on rigid billiards. But, beside the usual coordinates describing static billiards (for the impact point on the boundary and the direction of the movement), one must also introduce the energy and time. If, for each fixed time, the frozen boundary is a strictly convex plane curve, the associated time-dependent billiards can be modeled by a 4-dimensional, volume preserving diffeomorphism which maps successive impacts with the moving boundary.

In the special case of the breathing circle, the angular momentum is conserved, implying a reduction of the model to a 2-dimensional diffeomorphism. Then, using a corollary of Herman's *Théorème des Courbes Translatées* [4] we prove that if the motion of the moving boundary is sufficiently smooth then, for any admissible initial condition, the particle will move with bounded velocity and therefore will have a bounded gain of energy.

A similar case was studied by Levi in ([6] §1), where he shows examples of *pulsating soft circular billiards* whose energies stay bounded for all time.

The study of time-dependent billiards is motivated by two main questions. From one side, it is a natural extension to higher dimensions of the Fermi Accelerator, studied for instance in [4], [7] and [8]. From

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another side, among many branches of physics, it is of special interest on the study of the motion of particles inside collectively excited nuclei. On the attempts to understand the origin of dissipation of collective motion in finite Fermi systems (see, for instance, [2] and [3]) models were constructed where the classical part corresponds to time-dependent billiards, especially with multipole resonances (with the breathing circle corresponding to the monopole mode).

## 2 The breathing circle billiard map

Let $R(t)$ be a strictly positive $T$-periodic $C^k$ function, $k > 2$ and $\Gamma(t) = \{x^2 + y^2 = R^2(t)\}$ be the (moving) boundary of the breathing circle. Suppose that at an instant $t_n$ a particle is on $\Gamma(t_n)$ at a point $(x_n, y_n)$. Let the unitary tangent vector at that point be $\hat{\tau}_n$ and the unitary outward normal vector be $\hat{\eta}_n$. The point on $\Gamma(t_n)$ is then given by $(x_n, y_n) = \vec{R}_n = R_n\,\hat{\eta}_n = R(t_n)\,\hat{\eta}_n$. Suppose that it leaves the circle from $(x_n, y_n)$ with velocity $\vec{v}_n$.

It travels on a straight line and hits the boundary again at an instant $t_{n+1}$ at $(x_{n+1}, y_{n+1}) = \vec{R}_{n+1}$ with

$$\vec{R}_{n+1} = \vec{R}_n + (t_{n+1} - t_n)\vec{v}_n \tag{1}$$

and will then rebound with velocity

$$\vec{v}_{n+1} = \vec{v}_n - 2 < \vec{v}_n, \hat{\eta}_{n+1} > \hat{\eta}_{n+1} + 2\, \frac{dR(t)}{dt}\bigg|_{t=t_{n+1}} \hat{\eta}_{n+1}\ . \tag{2}$$

Let $\vec{L} = \vec{v} \times \vec{R}$ denote the angular momentum with respect to the center of the breathing circle, then

$$\vec{L}_{n+1} = \vec{v}_{n+1} \times \vec{R}_{n+1} = \vec{v}_n \times \vec{R}_{n+1} = \vec{v}_n \times \vec{R}_n = \vec{L}_n\ .$$

This shows that the angular momentum is conserved at impacts with the moving boundary. As it is also obviously conserved between impacts (since the particles moves on a straight line with constant velocity), angular momentum is conserved along all the movement. A first easy consequence of this conservation is that a particle never changes orientation with respect to the breathing circle, i.e., it always rotates in the same direction. In other words, if we introduce the notation $L_n = R_n\vec{v}_n \cdot \hat{\tau}_n$, is is easy to verify that $L_n$ is conserved.

Another consequence of the conservation of angular momentum is that a particle never stops if its angular momentum is different from zero. This is because if $v_n = |\vec{v}_n|$, as $L_n = L_0 \neq 0$ then $R_n v_n \geq |R_n\vec{v}_n \cdot \hat{\tau}_n| = |L_0|$ and $v_n \geq \frac{|L_0|}{R_n} \geq \frac{|L_0|}{\overline{R}}$, where $\overline{R} = \max\{R(t), t \in [0, T)\}$ .

It also follows that the traveling time of a particle (between two impacts with the moving boundary) is bounded, because

$$t_{n+1} - t_n = \frac{\text{distance}}{\text{velocity}} \leq \frac{2\overline{R}}{v_n} \leq \frac{2\overline{R}^2}{L_0} \tag{3}$$

for every $n$.

We introduce the variable $I_n = -R_n\vec{v}_n \cdot \hat{\eta}_n$. It is, in a certain sense, a natural variable, because it is the normal counterpart of the conserved quantity $L_n$.

For given $L_0 \neq 0$ and $I_0$, $t_0$ initial impact variables, let $I_1$ and $t_1$ be the values of $I$ and $t$ at the next impact with the breathing circle. Let $w_n = \frac{d}{dt}R^2(t)|_{t=t_n}$

**Lemma 1** *For any fixed $L_0 \neq 0$ and for every $(I_0, t_0)$ with $I_0 + \frac{w_0}{2} > 0$, the map $\mathcal{M} : (I_0, t_0) \mapsto (I_1, t_1)$ is defined by the formulae:*

$$I_1 = I_0 - w_1 + \left(\frac{L_0^2 + I_0^2}{R_0^2}(t_1 - t_0) - 2I_0\right) \tag{4}$$

$$(t_1 - t_0)\left(\frac{L_0^2 + I_0^2}{R_0^2}(t_1 - t_0) - 2I_0\right) = R_1^2 - R_0^2 \tag{5}$$

**Proof:** Equation (5) above is easily obtained after computing $|\vec{R}_1|^2$ from (1). Now, if the particle moves inside the breathing circle region, it's normal velocity at the impacts must be bigger than the normal velocity of the boundary, i.e., $-\vec{v}_0 \cdot \hat{\eta}_0 + R'(t_0) > 0$ or, using the variables introduced above, $I_0 + \frac{w_0}{2} > 0$. This condition implies that (5) has a solution $t_1 > t_0$ and thus the map is well defined under this assumption. (In [5] we have in fact proved a little more: $I_0 + \frac{w_0}{2} > 0$ implies $I_1 + \frac{w_1}{2} \geq 0$.)

On the other hand, from (1) we have that $(t_1 - t_0)\vec{v}_0 \cdot \vec{v}_0 = (\vec{R}_1 - \vec{R}_0) \cdot \vec{v}_0$. This relation together with the rebound formula (2) yields (4). ∎

Formulae (4) and (5) above are invariant under the translations $t + nT$, where $T$ is the period of $R(t)$ and $n$ is any integer. So, we can take $t \,(\mathrm{mod}\ T) \in S^1$ and then the domain of $\mathcal{M}$ is contained in the cylinder $I\!R \times S^1$. It is worthwhile to point out that, although time-dependent billiards are described by four variables, the breathing circle only takes two, namely $I$ and $t \,(\mathrm{mod}\ T)$, the other two independent variables being the (conserved) angular momentum and its conjugate variable. Moreover, as $L_0$ appears only squared in the calculations, we can suppose $L_0 > 0$. The case $L_0 = 0$ corresponds to the movement on the diameter of the breathing circle and is equivalent to the 1-dimensional Fermi accelerator, studied, for instance, in [4], [7] and [8].

Now let $(I_1, t_1) = \mathcal{M}(I_0, t_0)$ and define $(I_{-1}, t_{-1})$ by $\mathcal{M}(I_{-1}, t_{-1}) = (I_0, t_0)$. Let

$$\begin{aligned} A_1 &= \{\, (I_0, t_0) \mid I_0 + \frac{w_0}{2} > 0 \ \text{ and } I_1 + \frac{w_1}{2} = 0 \,\} \\ A_{-1} &= \{\, (I_0, t_0) \mid I_0 + \frac{w_0}{2} > 0 \ \text{ and } I_{-1} + \frac{w_{-1}}{2} = 0 \,\}. \end{aligned}$$

(With our assumptions on $R(t)$, the Lebesgue measure on $I\!R^2$ of these sets is zero.)

Let $D = I\!R \times S^1 \setminus (A_1 \cup A_{-1})$.

**Lemma 2** *If $R(t)$ is a strictly positive $T$-periodic $C^k$ function, $k > 2$ then for each fixed $L_0 \neq 0$ the map $\mathcal{M} : D \to D$, given by (4) and (5), is a $C^{k-2}$ diffeomorphism, preserving the area $d\mu = \dfrac{2I + w}{2R^2} dIdt$.*

**Proof:** the proof follows from Theorem 1 in [5] ∎

## 3 Changing coordinates: Some properties of the billiard map

From the section above, it seems natural to introduce the variable $J = 2I + w$. The advantages of this choice will become clear in what follows. In the new coordinates $(J, t)$ lemma 1 rewrites:

**Lemma 1'** *For a fixed $L_0 \neq 0$ and given an initial condition $(J_0, t_0)$ with $J_0 > 0$, its iterated by the billiard map, $(J_1, t_1)$, is given by the formulae:*

$$J_1 = J_0 - (w_0 + w_1) + 2\frac{R_1^2 - R_0^2}{t_1 - t_0} \tag{6}$$

$$(t_1 - t_0)\frac{4L_0^2 + (J_0 - w_0)^2}{4R_0^2} = J_0 - w_0 + \frac{R_1^2 - R_0^2}{t_1 - t_0}. \tag{7}$$

Since $R(t)$ is at least $C^2$, there exist $\xi_0$ and $\xi_1$ such that

$$R_1^2 - R_0^2 = \frac{w_0 + w_1}{2}(t_1 - t_0) + \frac{w'(\xi_0)}{4}(t_1 - t_0)^2 - \frac{w'(\xi_1)}{4}(t_0 - t_1)^2, \tag{8}$$

and

$$J_1 = J_0 + \frac{w'(\xi_0) + w'(\xi_1)}{2}(t_1 - t_0).$$

For $L_0 \neq 0$ and by inequality (3) we have

$$|J_1 - J_0| \leq \overline{w}'\frac{2\overline{R}^2}{|L_0|} \tag{9}$$

where $\overline{w}' = \max\{|w'(t)|, t \in [0, T]\}$.

**Remark**: Suppose that the breathing circle moves slowly, i.e., $R^2(t) = 1 + \epsilon g(t)$, for a small parameter $\epsilon$ and $g$ a $C^k$, $T$-periodic function. Then $w'(t) = \epsilon g''(t) = \mathcal{O}(\epsilon)$. Let $J_n$ be the value of $J$ at the n$^{th}$ impact. For $n = \mathcal{O}(\frac{1}{\epsilon})$

$$|J_n - J_0| \leq n\overline{w}'\frac{2\overline{R}^2}{|L_0|} = \mathcal{O}(1)$$

which shows that $J$ is an adiabatic invariant (see [1] for the definition of adiabatic invariant).

Let $B_\lambda$ denotes the cylinder $(\lambda, +\infty) \times S^1 = \{(J, t), J > \lambda, t \in S^1\}$.

**Lemma 3** *Suppose that $R(t)$ is a strictly positive $T$-periodic $C^k$ function, $k > 2$. Let*

$$M : B_{\overline{w}'(\frac{2\overline{R}^2}{|L_0|}+1)} \longrightarrow M(B_{\overline{w}'(\frac{2\overline{R}^2}{|L_0|}+1)}) \subset B_{\overline{w}'}$$

*be defined by formulae (6) and (7). Then $M$ is a $C^{k-2}$ diffeomorphism, preserving the measure $d\mu = \frac{J}{4R^2}dJdt$.*

**Proof:** For a fixed $L_0 \neq 0$, we have automatically from (9) that

$$J_0 > \overline{w}'(\frac{2\overline{R}^2}{|L_0|} + 1) \Rightarrow J_1 > \overline{w}' > 0 \ .$$

The proof follows immediately from this fact and lemma 2. ∎

**Lemma 4** *Given any curve* $\Gamma$ *in the cylinder* $B_{\overline{w}'(\frac{2\overline{R}^2}{|L_0|}+1)}$ *homotopic to* $S^1 \times \{0\}$ *then* $M(\Gamma) \cap \Gamma \neq \emptyset$*. In other words,* $M$ *has the property of intersection.*

**Proof:** $M$ preserves a measure, absolutely continuos with respect to the Lebesgue measure. This implies the property of intersection [4]. ∎

# 4 Invariant Curves

The purpose of this section is to investigate the behaviour of the breathing circle billiard map in the neighbourhood of $\infty$. We shall show that a map can be defined which exhibits some properties of twist maps. More specifically, using a result proved by R. Douady [4] we will prove the existence of invariant curves for values of $J$ sufficiently large.

For a fixed $L_0 \neq 0$, and since

$$J_0 > \overline{w}'(\frac{2\overline{R}^2}{|L_0|} + 1) \Rightarrow J_1 > \overline{w}' > 0$$

the change of variables $(z,t) = (\frac{1}{J}, t)$ is well defined on $B_{\overline{w}'(\frac{2\overline{R}^2}{|L_0|}+1)}$ and is $C^\infty$ .

Let $0 < \delta \leq \frac{|L_0|}{\overline{w}'(2\overline{R}^2+|L_0|)}$. Then the map

$$\begin{array}{rccc} N: & (0,\delta)\times S^1 & \rightarrow & I\!R^+ \times S^1 \\ & (z_0,t_0) & \mapsto & M(1/z_0,t_0) \end{array}$$

defined by

$$z_1 = z_0 \frac{1}{1+g(t_1,t_0)z_0} \text{ with } g(t_1,t_0) = -(w_0+w_1) + 2\frac{R_1^2 - R_0^2}{t_1 - t_0} \tag{10}$$

$$\left[4L_0^2 z_0^2 + (1-w_0 z_0)^2\right](t_1 - t_0) + 4R_0^2 z_0 (w_0 z_0 - 1) = 4R_0^2 z_0^2 \frac{R_1^2 - R_0^2}{t_1 - t_0} \tag{11}$$

is a $C^{k-2}$ diffeomorphism, with the property of intersection.

**Lemma 5** $N$ *can be extended to a* $C^{k-2}$ *diffeomorphism* $\tilde{N}$ *on a neighbourhood of* $\{0\} \times S^1$*.* $\tilde{N}$ *has the property of intersection and a development of the form:*

$$\begin{aligned} z_1 = &\quad z_0 + \mathcal{O}(z_0^2) \\ t_1 = &\quad t_0 + 4R_0^2 z_0 + \mathcal{O}(z_0^2) \end{aligned}$$

**Proof:** First of all, let us show that $t_1 = t_1(t_0, z_0)$ can be extended for $z_0 < 0$. $R(t)$ being $C^k$, $k > 2$ then $f(t_1 - t_0) = R_1^2 - R_0^2 - w_0(t_1 - t_0)$ is a $C^{k-1}$ function with $f(0) = f'(0) = 0$. By Hadamard's Lemma $f(t_1 - t_0) = (t_1 - t_0)h(t_1 - t_0)$, where $h$ is a $C^{k-2}$ function. Using (11) and discarding the trivial solution $t_1 \equiv t_0$, we have:

$$\left[4L_0^2 z_0^2 + (1-w_0 z_0)^2\right](t_1 - t_0) + 4R_0^2 z_0 (w_0 z_0 - 1) = 4R_0^2 z_0^2 \left[w_0 + h(t_1 - t_0)\right].$$

Let

$$F(t_0,t_1,z_0) = \left[4L_0^2z_0^2 + (1-w_0z_0)^2\right](t_1-t_0) + 4R_0^2z_0(w_0z_0-1) - 4R_0^2z_0^2\left[w_0 + h(t_1-t_0)\right].$$

$F$ is a $C^{k-2}$ function on $S^1\times S^1\times I\!R$ such that $F(t_0,t_0,0)=0$ and $\frac{\partial F}{\partial t_1}(t_0,t_0,0)=1$. If $k>2$ and for each fixed $t_0\in S^1$, by the Implicit Function Theorem we have $t_1=t_1(z_0,t_0)$ on a neighbourhood of $(0,t_0)$. Moreover, as $S^1$ is compact and $t_1(0,t_0)=t_0$ for every $t_0$, we can find a neighbourhood of $\{0\}\times S^1$ where $t_1=t_1(z_0,t_0)$ is a $C^{k-2}$ function.

This implies that $g(z_0,t_0) = -(w(t_0)+w(t_1(z_0,t_0))) + 2\frac{R^2(t_1(z_0,t_0))-R^2(t_0)}{t_1(z_0,t_0)-t_0}$ has a $C^{k-2}$ extension on a neighbourhood of $\{0\}\times S^1$. Clearly it is bounded and, perhaps on a smaller neighbourhood, we can take $|g(t_1,t_0)z_0|<1$. So, $z_1(z_0,t_0)$ has a $C^{k-2}$ extension on a neighbourhood of $\{0\}\times S^1$.

Analogously, $N^{-1}$ can be extended on a neighbourhood of $\{0\}\times S^1$. $N$ is then extended to a $C^{k-2}$ diffeomorphism $\tilde{N}$ on a neighbourhood of $\{0\}\times S^1$.

Since the formulae remain unchanged, the property of intersection is preserved for curves on $z_0>0$ or $z_0<0$. But $\tilde{N}(0,t_0)=(0,t_0)$ for any $t_0\in S^1$. Then, if a curve $\Gamma$ cuts the circle $\{0\}\times S^1$, $\tilde{N}(\Gamma)\cap\Gamma\neq\emptyset$, and $\tilde{N}$ has the property of intersection.

Finally, a simple calculation leads to the development of $\tilde{N}$. ∎

**Proposition 6** *If $k>7$, we have a family of invariant curves for $\tilde{N}$ that protect $\{0\}\times S^1$, in the following sense: Every orbit of $\tilde{N}$ with no intersection with $\{0\}\times S^1$, stay at a bounded distance, above and bellow, from $\{0\}\times S^1$. Those bounds are arbitrarily small strictly positive constants.*

**Proof:** Lemma 5 give the conditions to apply a corollary of Herman's *Théorème des Courbes Translatées*, proved by R. Douady ( [4], page III - 8) and the result follows. ∎

# 5 Boundedness of the velocity

Coming back to the coordinates $(J,t)$ and the diffeomorphism $M$, let us denote by $(J_n,t_n)$ the points of the orbit of $(J_0,t_0)$ under $M$. An admissible initial condition for $M$ is the set of $(J_0,t_0)$ such that $J_n>0,\forall n$ (which is the condition of existence of the next impact given in lemma 1').

**Theorem 1** *Given the billiard on the breathing circle $x^2+y^2=R(t)^2$, with $R(t)$ a strictly positive $T$-periodic $C^k$ function, $k>7$, then, for any admissible initial condition, a particle will move with bounded velocity.*

**Proof:** For $L_0\neq 0$, since $\tilde{N}$ has a family of invariant curves that *protect* $\{0\}\times S^1$ and $\tilde{N}=N$ for $z_0>0$, we have a family of invariant curves of $M$ that *protect* $\infty$, i.e., there exist a family of functions

$\Gamma_n : S^1 \to (\overline{w}'(\frac{2\overline{R}^2}{|L_0|} + 1), +\infty)$, approaching infinity with norm $C^{k-2}$, whose graphs are invariant under $M$.

Given $(J_0, t_0)$ such that $\Gamma_l(t_0) < J_0 < \Gamma_m(t_0)$ for some $l$ and $m$, we will have that $0 < \inf_{t \in S^1} \Gamma_l(t) < J_n < \sup_{t \in S^1} \Gamma_m(t)$, for every $n$. Therefore $(J_0, t_0)$ is an admissible initial condition and $J_n$ will remain bounded.

On the other hand, if $(J_0, t_0)$ is bellow all the invariant curves, then $J_n$ will be bounded above although it may be equal to zero. Anyway, for every admissible initial condition, $J_n$ will remain bounded.

Since $J_n = 2(I_n + \frac{w_n}{2}) = 2(-R_n \vec{v}_n \cdot \hat{\eta}_n + \frac{w_n}{2})$, it is obvious that $J_n$ is bounded if and only if $v_n = |\vec{v}_n|$ is bounded.

If $L_0 = 0$, we have the motion along the diameter of the breathing circle, that corresponds to the 1-dimensional Fermi accelerator, which has bounded velocities for any admissible initial condition (see, for instance, [4]). ■

**Remark**: This result is also true if we consider the billiard in any higher dimension breathing sphere $S^n$, instead of the circle $S^1$, since the conservation of the angular momentum implies that the movement will occur on a plane and so, the effective boundary will be the breathing circle.

In terms of the dynamics of the breathing circle billiard, we can translate Theorem 1 in the following way: For each fixed $L_0$, there exists a bound $K$, depending on $R(t)$ and decreasing with $L_0$, such that for $J > K$, there are infinitely many spanning curves that may not exist for $J < K$. This fact is illustrated on the pictures bellow where we plot the phase-space for $R(t) = 1 + \frac{0.1}{4\pi^2}\cos 2\pi t$ and different choices of $L_0$. (Because of the special form of $R(t)$, $J$ is, in fact, an eternal adiabatic invariant and, for big $J$'s, all curves are invariant.)

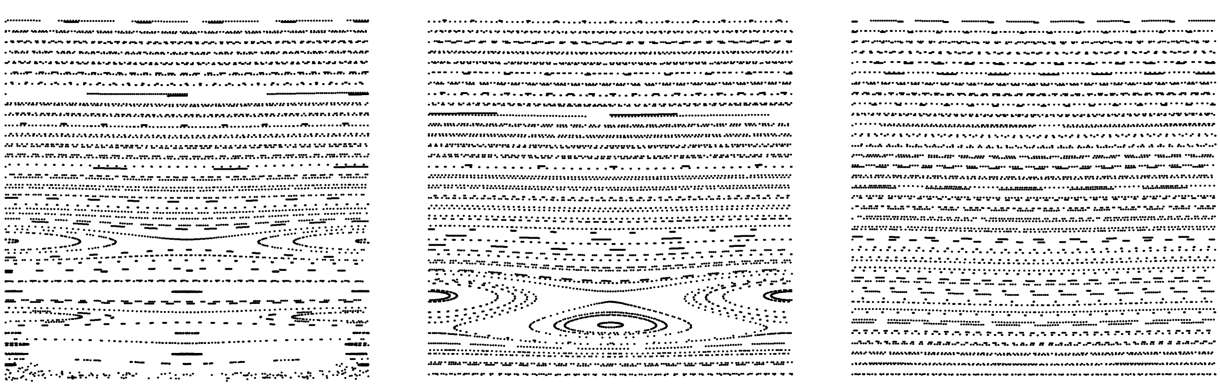

Figure 1: Phase-space of the breathing circle for $t \in [0, 1)$, $J \in [0, 8]$, and $L_0 = 0.25, 0.98$ and $1.3$.

Clearly our approach does not apply to the low energy regime (i.e., for small $L_0$'s and/or small $J$'s). For the 1-dimensional case, a simplified model was introduced by Lichtemberg and Liebermann [7]. This approximation and some comparisons with the standard map seems to give some valuable insights about this energy regime. In the simplified model one allows the boundary to interact with the particle through

momentum exchange (via a given $w(t)$ periodic in time), but assuming that the boundary does not change in time and so $R(t) \equiv R_0$. In this case, the associated billiard map is given by

$$
\begin{aligned}
J_1 &= J_0 - (w_0 + w_1) \\
t_1 &= t_0 + \frac{4R_0^2}{4L_0^2 + (J_0 - w_0)^2}(J_0 - w_0).
\end{aligned}
$$

This simplified model allows unphysical negative values of $J$, which are forbidden in the complete model.

On the next set of pictures we show the phase-space of the simplified model with the same conditions and $w(t)$ we took in figure 1. Except for a small neighbourhood of $J = 0$, the dynamical behaviour seems astonishingly similar.

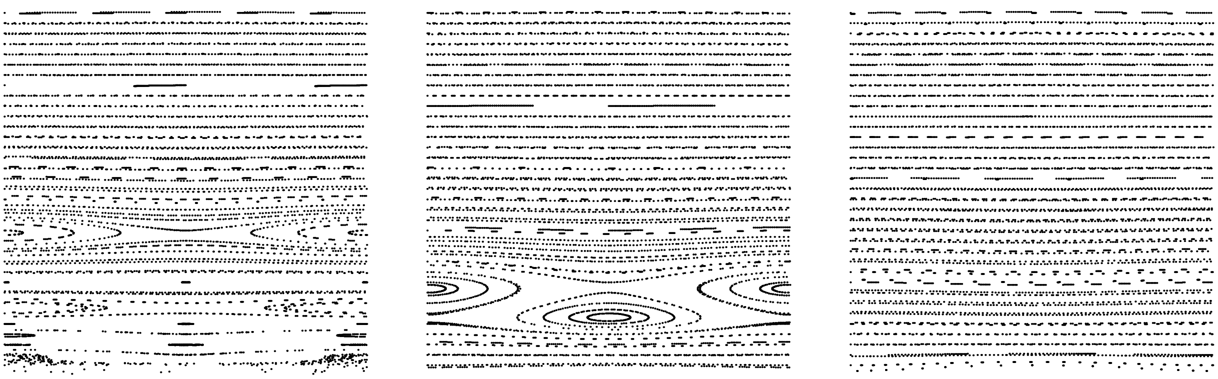

Figure 2: Phase-space of the simplified breathing circle for $t \in [0, 1)$, $J \in [0, 8]$, and $L_0 = 0.25, 0.98$ and 1.3.

Finally we point out that the existence of a conserved quantity (the angular momentum) is the clue of our proof, since it reduces the dimension to two. Unfortunately, it is easy to prove that a 2-dimensional moving billiard, with convex frozen boundary and with only normal deformation, preserves the angular momentum if and only if it is the breathing circle. So, the extension of this result to other time dependent billiards will depend on the existence of some constant of motion or to the extension of the techniques to higher dimensions.

# References


[1] V. I. Arnold: Small denominators and problems of stability of motion in classical and celestial mechanics, Russ. Math. Surveys **18:6**, 85-191 (1963).

[2] J. Blocki, Y. Boneh, J. R. Nix, J. Randrup, M. Robel, A. J. Sierk, W. J. Swiatecki: One-Body Dissipation and the Super-Viscidity of Nuclei, Annals of Physics **113**-2, 330-386 (1978).

[3] G. F. Burgio, M. Baldo, A. Rapisarda: Chaoticity in Vibrating Nuclear Billiards, Phys.Rev.C, **52**, 2475- (1995).

[4] R. Douady: Applications du théorème des tores invariants, Thèse de 3ème Cycle, Univ. Paris VII (1982).

[5] J. Koiller, R. Markarian, S. Oliffson Kamphorst, S. Pinto de Carvalho: Time-dependent billiards, Nonlinearity **8**, 983-1003 (1995).

[6] M. Levi: Quasiperiodic motions in superquadratic time-periodic potentials, Comm. Math. Phys, **143** 43-83 (1991).

[7] A.J. Lichtenberg, M.A. Liebermann: *Regular and Stochastic Motion*, Springer-Verlag Appl. Math. Sci. **38**, Berlin, Heidelberg, New York, Springer-Verlag (1983).

[8] T. Krüger, L.D. Pustyl'nikov, S.E. Troubetzkoy: Acceleration of bouncing balls in external fields, Nonlinearity, **8**, 397-410 (1995).
L.D. Pustyl'nikov: On Ulam's Problem, Theor. Math. Phys. **57**, 1035-1038 (1983).
L.D. Pustyl'nikov: On the Fermi-Ulam Model, Soviet Math. Dokl. **35:1**, 88-92 (1987).